

RESOLUTIONS OF MODULES WITH INITIALLY LINEAR SYZYGIES

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ABSTRACT. We introduce the class of modules with initially linear syzygies, a class including ideals with linear quotients, and study their minimal resolutions. Using a contracting homotopy for the resolutions, we see that the minimal resolution of a matroidal monomial ideal admits a DGA structure.

1. INTRODUCTION

Let k be a field and $S = k[x_1, \dots, x_n]$, the polynomial ring over k . In this paper we introduce the notion of modules with initially linear syzygies, and study the structure of their minimal resolutions. We show how to construct the minimal resolution of a module with initially linear syzygies using discrete Morse theory, and we then show that such modules are componentwise linear; finally, at the end of the paper we investigate multiplicative properties, and we show that the minimal free resolution of S/I admits a differential graded algebra structure, where I is either a stable monomial ideal or a squarefree matroidal ideal. The result for stable ideals is not new, it has been shown by Peeva [Pee96], but we obtain a simpler proof. The result for squarefree matroidal ideals is new, however. We finish the paper by calculating the product on a part of the minimal resolution of the ideal coming from the Fano matroid as an illustration of the results.

2. MODULES WITH INITIALLY LINEAR SYZYGIES

Definition 1. A presentation of a finitely generated graded S -module M ,

$$(1) \quad 0 \longrightarrow \operatorname{Ker} \eta \longrightarrow \bigoplus_i S \cdot g_i \xrightarrow{\eta} M \longrightarrow 0,$$

is said to have *initially linear syzygies* with respect to a term order \prec on the free module $\bigoplus_i S \cdot g_i$, if $\operatorname{Ker} \eta \subseteq \bigoplus_i \mathfrak{m} \cdot g_i$, and if the initial module $\operatorname{in}_{\prec}(\operatorname{Ker} \eta)$ is generated by terms of the form $x_j g_i$.

We will say that M has initially linear syzygies if it has such a presentation for some choice of generating set $\{g_1, \dots, g_n\}$ and term order \prec .

Ideals with initially linear syzygies generalise ideals with linear quotients; let us recall that an ideal I has linear quotients if there are elements f_1, f_2, \dots, f_n such that $I = (f_1, \dots, f_n)$, and for each $1 < i \leq n$, the colon ideal $(f_1, \dots, f_{i-1}) : f_i$ is generated by linear forms. It is not hard to see that a homogeneous ideal has linear quotients if and only if it has initially linear syzygies with respect to a position over term order, which is a term order on a free module $\bigoplus_i S \cdot g_i$ such that $g_i < g_j$ implies

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that $x^\alpha g_i < x^\beta g_j$ for all α and β . Ideals with linear quotients in turn generalise shellable monomial ideals, a concept introduced by Batzies and Welker [BW02]. A monomial ideal I with minimal monomial generators m_1, \dots, m_t is *shellable* if there is a total order “ \sqsubset ” on m_1, \dots, m_t such that for m_j, m_i with $m_j \sqsubset m_i$ there is an m_k such that $m_k \sqsubset m_i$ and $x_{g(m_k, m_i)} m_i = \text{lcm}(m_k, m_i)$ divides $\text{lcm}(m_j, m_i)$ for some index $g(m_k, m_i)$. Thus a shellable monomial ideal has linear quotients and then also initially linear syzygies.

The families of monomial ideals we are particularly interested in are the stable ideals, and the squarefree matroidal ideals, which are ideals where the supports of the minimal generators form the bases of a matroid.

Since we will extensively use algebraic Morse theory, we will below briefly review our terminology. For details on algebraic Morse theory, see [JW09], [Jon03], [Koz05] and [Skö06]. By a *based complex* of R -modules we mean a chain complex K_\bullet of R -modules together with a direct sum decomposition $K_n = \bigoplus_{\alpha \in I_n} K_\alpha$ where $\{I_n\}$ is a family of mutually disjoint index sets. For $f : \bigoplus_n K_n \rightarrow \bigoplus_n K_n$ a graded map, we write $f_{\beta, \alpha}$ for the component of f going from K_α to K_β , and given a based complex K_\bullet we construct a digraph Γ_{K_\bullet} with vertex set $V = \bigcup_n I_n$ and with a directed edge $\alpha \rightarrow \beta$ whenever the component $d_{\beta, \alpha}$ is non-zero.

A *partial matching* on a digraph $D = (V, E)$ is a subset A of the edges E such that no vertex is incident to more than one edge in A . In this situation we define the new digraph $D^A = (V, E^A)$ to be the digraph obtained from D by reversing the direction of each arrow in A . Given the matching A , we define the sets A^+ , A^- and A^0 by letting A^+ be the set of vertices that are targets of a reversed arrow from A ; A^- be the set of vertices that are sources of a reversed arrow from A ; and A^0 to be the vertices that are not incident to an arrow from A . We call a partial matching A supported on the digraph Γ_{K_\bullet} a *Morse matching* if, for each edge $\alpha \rightarrow \beta$ in A , the corresponding component $d_{\beta, \alpha}$ is an isomorphism, and furthermore there is a well founded partial order \prec on each I_n such that $\gamma \prec \alpha$ whenever there is a path $\alpha^{(n)} \rightarrow \beta \rightarrow \gamma^{(n)}$ in $\Gamma_{K_\bullet}^A$.

3. THE MINIMAL RESOLUTION

In this section we start by observing that a finitely generated S -module M has a free resolution given by a two-sided Koszul complex G_\bullet . The modules in this resolution are not even finitely generated, so it is far from being minimal. In this resolution we see that we can find a matching that gives us a projection that allows us to find the minimal resolution F_\bullet of M as a direct summand of G_\bullet in the case when M has initially linear syzygies. We then give a description of the differential in terms of reductions following Jöllenbeck and Welker [JW09], and then show that in some cases the differential is of Eliahou–Kervaire type. We round off by showing that modules with initially linear syzygies are componentwise linear.

Let M be a finitely generated S -module, let V be the k -vector space with basis e_1, \dots, e_n and let F_\bullet be the chain complex with modules $F_p = S \otimes \text{Alt}^p V \otimes M$. For an element $e_{i_1} \wedge \dots \wedge e_{i_d}$ of $\text{Alt}^d V$ with $I = \{i_1, \dots, i_d\} \subseteq [n]$ and $i_1 < \dots < i_d$, we will write e_I , and we will also write $e_I m$ for the element $1 \otimes e_I \otimes m$; as m ranges over a k -basis of M , these elements obviously form an S -basis for F_n . The differential $d_n : S \otimes \text{Alt}^n V \otimes M \rightarrow S \otimes \text{Alt}^{n-1} V \otimes M$ is defined on the basis elements by

$$d(e_I m) = \sum_{i \in I} \varepsilon(i; I) (x_i e_{I \setminus i} m - e_{I \setminus i} x_i m),$$

where the sign $\varepsilon(i; I)$ is defined by

$$\varepsilon(i; I) = (-1)^{|\{j \mid j \in I, j < i\}|}.$$

Lemma 1. *The complex F_\bullet is a free resolution of M .*

Proof. It is obvious that $H_0(F_\bullet) \simeq M$, so we just have to prove that $H_i(F_\bullet) = 0$ for $i \geq 1$.

For a k -vector space basis B of M we consider F_\bullet as a based complex of k -vector spaces via the natural decomposition

$$S \otimes \text{Alt}^\bullet V \otimes M \simeq \bigoplus_{\alpha \in \mathbb{N}^n, I \subseteq [n], m \in B} k \cdot x^\alpha e_I m.$$

For each i we now define the following subset of the vertices in the digraph Γ_{F_\bullet} :

$$V_i = \{x^\alpha e_I m \mid \deg x^\alpha + |I| = i\}.$$

Now, construct a partial matching E_i on the subgraph $\Gamma_{F_\bullet}|_{V_i}$ consisting of the edges

$$E_i = \{x^\alpha e_I m \rightarrow x^\alpha x_i e_{I \setminus i} m \mid i = \min(\text{supp } \alpha \cup I)\}.$$

It should be clear that if $\alpha \rightarrow \beta \in E_i$ for $\alpha, \beta \in V_i$ then all $\gamma \in V_i$, $\gamma \neq \beta$ with an edge $\alpha \rightarrow \gamma$ are unmatched, so E_i is a Morse matching on $\Gamma_{F_\bullet}|_{V_i}$, and since for all edges $\alpha \rightarrow \beta$ with $\alpha \in V_i$ and $\beta \in V_j$ we have $j \leq i$, we get from [Skö06, Lemma 7] that $\bigcup_i E_i$ is a Morse matching on Γ_{F_\bullet} . The claim of the lemma now follows from [Skö06, Theorem 1] since there are no E -critical vertices in degree 1 and higher. \square

Now, we will construct a matching on G_\bullet , viewed as a based complex of S -modules, that will give us a splitting homotopy φ of G_\bullet . Using the homotopy φ we can then describe the minimal resolution of M . Given a presentation (1), we define for a basis element g_i of the free module $\bigoplus_j S \cdot g_j$, its critical and non-critical indices by

$$\text{crit}(g_i) = \{j \mid x_j g_i \in \text{in}_<(\text{Ker } \eta)\}, \quad \text{ncrit}(g_i) = [n] \setminus \text{crit}(g_i).$$

Suppose M has initially linear syzygies with a presentation as in (1), we then have a k -basis for M , (abusing notation by writing $x^\alpha g_i$ for $\eta(x^\alpha g_i)$),

$$\{x^\alpha g_i \mid \text{supp } \alpha \subseteq \text{ncrit } g_i\}.$$

We consider the resolution F_\bullet as a based complex of S -modules via the decomposition

$$F_n \simeq \bigoplus_{\substack{I, \alpha, i \\ |I|=n, \text{supp } \alpha \subseteq \text{ncrit } g_i}} S \cdot e_I x^\alpha g_i.$$

For each term m in $\bigoplus_i S \cdot g_i$, we define a subset of the vertices of Γ_{F_\bullet} .

$$V_m = \{S \cdot e_I x^\alpha g_j \mid x_I x^\alpha g_j = m\},$$

and for each such m , we will now define a partial matching E_m on the digraph $\Gamma_{F_\bullet}|_{V_m}$ by

$$E_m = \{S \cdot e_I x^\alpha g_j \rightarrow S \cdot e_{I \setminus i} x_i x^\alpha g_j \mid i = \max((I \cup \text{supp } \alpha) \cap \text{ncrit } g_j)\}.$$

Lemma 2. *The set $E = \bigcup_m E_m$ is a Morse matching on the digraph Γ_{F_\bullet} . The set of unmatched vertices consists of all $S \cdot e_I g_j$ where $I \subseteq \text{crit } g_j$.*

Proof. It is clear that E_m is a partial matching on $\Gamma_{F_\bullet}|_{V_m}$ and along the same lines as in the proof of Lemma 1, if $\alpha \rightarrow \beta \in E_m$ for $\alpha, \beta \in V_m$ then all $\gamma \in V_m$ such that $\gamma \neq \beta$ and with an edge $\alpha \rightarrow \gamma$ are unmatched. Thus, there is no oriented cycle in the finite digraph $\Gamma_{F_\bullet}^E|_{V_m}$, and by [Skö06, Lemma 1], E_m is a Morse matching on $\Gamma_{F_\bullet}|_{V_m}$. Whenever $\alpha \rightarrow \beta \in E$ with $\alpha \in V_{m_1}$ and $\beta \in V_{m_2}$ we have $m_2 \preceq m_1$, so by [Skö06, Lemma 7], $\bigcup_m E_m$ is a Morse matching on the full graph Γ_{F_\bullet} . \square

With the above result, we can now define an S -linear splitting homotopy φ on the resolution F_\bullet that allows us to construct a smaller resolution G_\bullet . We will first give an explicit recursive definition of φ . In the definition we use the following notation due to Knuth: When P is some proposition, then $[P] = 1$ if P is true, and $[P] = 0$ if P is false. Let us start by defining the two helper functions ι and φ_0 by

$$\iota(\alpha) = \max(\text{supp } \alpha)$$

$$\varphi_0(e_I x^\alpha g_j) = [\iota(\alpha) > \max(I \cap \text{ncrit } g_j)] \varepsilon(\iota(\alpha); I \cup \iota(\alpha)) \cdot e_{I \cup \iota(\alpha)} \frac{x^\alpha}{x_{\iota(\alpha)}} g_j$$

then

$$\varphi(e_I x^\alpha g_j) = \varphi_0(e_I x^\alpha g_j) - \varphi(d\varphi_0(e_I x^\alpha g_j) - e_I x^\alpha g_j).$$

Let π be the projection

$$\pi : \bigoplus_{I, \alpha, j} S \cdot e_I x^\alpha g_j \longrightarrow \bigoplus_{\substack{I, j \\ I \subseteq \text{crit}(g_j)}} S \cdot e_I g_j.$$

Now we can define a complex G_\bullet by letting

$$G_n = \bigoplus_{\substack{I, j \\ I \subseteq \text{crit}(g_j), |I|=n}} S \cdot e_I g_j$$

and defining the differential by $d_G = \pi(d_F - d_F \varphi d_F)$; we can then formulate the following result that generalises work of Batzies and Welker [BW02] and Herzog and Takayama [HT02].

Theorem 1. *Let M be a module with initially linear syzygies, then G_\bullet is the minimal free resolution of M .*

Proof. Follows from applying [Skö06, Theorem 2] to the matching E and the resolution F_\bullet . \square

From Theorem 1, we can now immediately deduce the following two corollaries giving the projective dimension and Castelnuovo–Mumford regularity of a module with initially linear syzygies.

Corollary 1. *If M is minimally generated by m_1, \dots, m_r , and M has initially linear syzygies then $\text{p.dim } M = \max_j |\text{crit } e_j|$*

Corollary 2. *If M is minimally generated by m_1, \dots, m_r , and M has initially linear syzygies then $\text{reg } M = \max_{i,j} (\deg m_i - \deg m_j)$*

By specialising the last corollary we get:

Corollary 3. *A graded module generated by homogeneous elements of the same degree with initially linear syzygies has a linear resolution.*

Another, non-recursive, way of describing the differential in F_\bullet is in terms of *reductions*, as used by Jöllenbeck and Welker [JW09] in their description of the differential in the Anick resolution. In the Morse digraph $\Gamma_{G_\bullet}^E$, we define an *elementary reduction path* to be a zig-zag path of length 2

$$\alpha_0 \rightarrow \beta \rightarrow \alpha_1,$$

where α_0, α_1 are in degree i , and β is in degree $i - 1$ if α_0 is in $E^0 \cup E^+$, and in degree $i + 1$ if $\alpha \in E^-$. To such a path we assign the corresponding *elementary reduction* which is the map

$$\rho_{\alpha_1, \alpha_0} = \begin{cases} -d_{\beta, \alpha_1}^{-1} \circ d_{\beta, \alpha_0}, & \text{if } \alpha_0 \in E^0 \cup E^+, \\ -d_{\alpha_1, \beta} \circ d_{\alpha_0, \beta}^{-1}, & \text{if } \alpha_0 \in E^-. \end{cases}$$

The matching condition implies that there is at most one elementary reduction path from α_0 to α_1 .

In general, we let a *reduction path* be a composition of zero or more elementary reduction paths

$$p = \alpha_0 \rightarrow \beta_1 \rightarrow \alpha_1 \rightarrow \cdots \rightarrow \beta_n \rightarrow \alpha_n, \quad n \geq 0,$$

and the corresponding reduction to be the composition

$$\rho_p = \rho_{\alpha_n, \alpha_{n-1}} \circ \rho_{\alpha_{n-1}, \alpha_{n-2}} \circ \cdots \circ \rho_{\alpha_1, \alpha_0}.$$

We denote the set of all reduction paths from α to β by $[\alpha \rightsquigarrow \beta]$. From [Skö06, Lemma 5] we can now conclude that for a basis element $e_I g_\alpha$, we can write

$$(2) \quad d_G(e_I g_\alpha) = \sum_{\substack{e_K g_\delta \\ K \subseteq \text{crit } g_\delta}} \sum_{\substack{e_J x^\beta g_\gamma \\ e_I g_\alpha \rightarrow e_J x^\beta g_\gamma}} \sum_{p \in [e_J x^\beta g_\gamma \rightsquigarrow e_K g_\delta]} \rho_p d_{e_J x^\beta g_\gamma, e_I g_\alpha}(e_I g_\alpha).$$

In our case we can see that we can divide the elementary reduction paths originating in a vertex in E^- into two types. We say that an elementary reduction path is of type 1 when it is of the form

$$e_I x^\alpha g_k \rightarrow e_{I \cup j} \frac{x^\alpha}{x_j} g_k \rightarrow e_{(I \cup j) \setminus i} \frac{x^\alpha}{x_j} g_k,$$

where j is the maximal element in $(\text{supp } \alpha \cup I) \cap \text{ncrit } g_k$. The corresponding reduction map is

$$\rho(e_I x^\alpha g_k) = \varepsilon(j; I \cup j) \varepsilon(i; I \cup j) x_i e_{(I \cup j) \setminus i} \frac{x^\alpha}{x_j} g_k.$$

An elementary reduction path is of type 2 if it is of the form

$$e_I x^\alpha g_k \rightarrow e_{I \cup j} \frac{x^\alpha}{x_j} g_k \rightarrow e_{(I \cup j) \setminus i} x^\beta g_l,$$

where $x^\beta g_l$ appears with nonzero coefficient $\lambda_{i,j,k,\alpha,\beta,l}$ in $\text{nf}((x_i x^\alpha / x_j) g_k)$, and j is the maximal element in $(\text{supp } \alpha \cup I) \cap \text{ncrit } g_k$. The corresponding reduction map is

$$\rho(e_I x^\alpha g_k) = -\varepsilon(j; I \cup j) \varepsilon(i; I \cup j) \lambda_{i,j,k,\alpha,\beta,l} e_{(I \cup j) \setminus i} x^\beta g_l.$$

We will now see that in well-behaved cases, there is a more explicit description of the differential.

Definition 2. If M has a presentation with initial linear syzygies such that for every generator g we have that

$$\text{crit}(\text{nf}(x_i g)) \subseteq \text{crit}(g),$$

we say that M is crit-monotone. (We interpret $\text{crit}(\sum_{j \in J} p_j e_j)$ as $\bigcup_{j \in J} \text{crit}(e_j)$ if there are no redundant terms in the sum.)

It is easy to see that stable monomial ideals are crit-monotone, and in [HT02], Herzog and Takayama prove that matroidal ideals are crit-monotone, and they show that the differential in the minimal resolution of crit-monotone monomial ideals is of Eliahou–Kervaire type. Below, we will generalise their result to general crit-monotone modules with initially linear syzygies.

For a basis element $e_I g$ with $I = \{i_1, \dots, i_n\}$ we define maps d_j^L, d_j^R , the left and right components of the differential, for $1 \leq j \leq n$ by

$$d_j^L(e_I g) = x_{i_j} e_{I \setminus i_j} g$$

and

$$d_j^R(e_I g) = \sum_k [I \setminus i_j \subseteq \text{crit}(g_k)] p_k e_{I \setminus i_j} g_k,$$

where $\text{nf}(x_{i_j} g) = \sum_k p_k g_k$ for $p_k \in S$ and g_k a basis element, and we again make use of the Knuth notation.

Theorem 2. *Let M be a crit-monotone module with initially linear syzygies, then the differential d in the minimal resolution G_\bullet is given in degree n by $d = \sum_{i=1}^n (-1)^{i-1} (d_i^L - d_i^R)$*

Proof. Consider an elementary reduction path, $e_I x^\alpha g_i \rightarrow e_J x^\beta g_j$, it is then easy to see that

$$|J \cap \text{crit } g_j| \leq |I \cap \text{crit } g_i|.$$

This observation together with the fact that we are only interested in reduction paths ending in vertices $e_J g$ with $J \subseteq \text{crit } g$, gives that the only non-trivial reduction paths appearing in the sum (2) are concatenations of elementary reduction paths of type 1 that are of the form

$$e_I x^\alpha g_k \rightarrow e_{I \cup j} \frac{x^\alpha}{x_j} g_k \rightarrow e_I \frac{x^\alpha}{x_j} g_k,$$

which have coefficient x_j in the corresponding reduction. The reduction paths of length 0 that appear in the sum (2) thus contribute $\sum_{i=1}^n (-1)^{i-1} d_i^L$ to the differential, and the concatenations of reduction paths mentioned above contribute $-\sum_{i=1}^n (-1)^{i-1} d_i^R$, which sums to the desired formula. \square

Herzog and Hibi have introduced the concept of componentwise linear ideals [HH99]. For a graded module M we let $M_{\langle j \rangle}$ be the module generated by all homogeneous elements of degree j in M ; using this notation we say that M is *componentwise linear* if the modules $M_{\langle j \rangle}$ have linear resolutions for all j .

Theorem 3. *If the finitely generated graded module M has initially linear syzygies, then $M_{\langle d \rangle}$ has initially linear syzygies for all d .*

Proof. By the hypothesis, there are homogeneous elements m_1, m_2, \dots, m_g , that generate M , and a presentation

$$0 \longrightarrow \text{Ker } \eta \longrightarrow \bigoplus_{i=1}^g S \cdot e_i \xrightarrow{\eta} M \longrightarrow 0$$

with $\eta(e_i) = m_i$, together with a term order ' \prec ' on $F = \bigoplus_{i=1}^g S \cdot e_i$ such that $\text{Ker } \eta$ has a Gröbner basis G consisting of initially linear terms.

We will now construct an explicit Gröbner basis G_d for the syzygies of $M_{\langle d \rangle}$, and we will start by constructing a presentation of $M_{\langle d \rangle}$. This module is minimally generated by the images of the elements $x^\alpha e_i$ of degree d in M that are irreducible with respect to G , and has a k -basis given by all the irreducible elements $x^\alpha e_i$ where $\deg e_i \leq d$. Thus, we consider the free module with basis elements $t^\alpha e_i$ where $\deg t^\alpha + \deg e_i = d$ and $x^\alpha e_i$ irreducible with respect to G , and define the map η_d by $\eta_d(t^\alpha e_i) = x^\alpha \eta(e_i) = x^\alpha m_i$. We now have the presentation

$$0 \longrightarrow \text{Ker } \eta_d \longrightarrow \bigoplus S \cdot t^\alpha e_i \xrightarrow{\eta_d} M_{\langle d \rangle} \longrightarrow 0,$$

where the direct sum ranges over all (α, i) such that $x^\alpha e_i$ is irreducible with respect to G and has degree d .

Now, we define the term order \prec_d on the free module $F_d = \bigoplus S \cdot t^\alpha e_i$ by letting $x^\alpha \cdot t^\beta e_i \prec_d x^\gamma \cdot t^\delta e_j$ if

$$\begin{aligned} x^{\alpha+\beta} e_i &\prec x^{\gamma+\delta} e_j, \text{ or} \\ x^{\alpha+\beta} e_i &= x^{\gamma+\delta} e_j, \text{ and } t^\beta \leq_{\text{revlex}} t^\delta. \end{aligned}$$

Let G_d be a set consisting of two types of elements. First, for every element $x_a \cdot e_i - \sum_j g_j \cdot e_j \in G$, and every monomial x^α of degree $d - \deg e_i$ such that $\text{supp } \alpha \subseteq \text{ncrit } e_i$ we consider $x_a x^\alpha e_i - \sum_j \sum_k h_{j,k} \cdot e_k$ such that $\text{nf}(x^\alpha g_j e_j) = \sum_k h_{j,k} \cdot e_k = \sum_{k,l} c_{j,k,l} x^{\beta_{j,k,l}} e_k$ and we let $x_a \cdot t^\alpha e_i - \sum_{j,k,l} c_{j,k,l} x^{\beta'_{j,k,l}} \cdot t^{\beta''_{j,k,l}} e_k \in G_d$, where $x^{\beta_{j,k,l}} = x^{\beta'_{j,k,l}} x^{\beta''_{j,k,l}}$ in such a way that $\deg x^{\beta''_{j,k,l}} e_k = d$. Second, for every e_i and every α such that $\text{supp } \alpha \subseteq \text{ncrit } e_i$ and $\deg x^\alpha = d - \deg e_i - 1$, we have the elements $x_a \cdot t_b t^\alpha e_i - x_b \cdot t_a t^\alpha e_i$ for each $a, b \in \text{ncrit}(e_i)$.

The claim is now that G_d is a Gröbner basis for $\text{Ker } \eta_d$. To start with, it is clear that G_d lies in the kernel of η_d , so it is sufficient to prove that for all $i \geq d$, there is a bijection between the terms of degree i in F_d which are irreducible with respect to G_d , and the terms of degree i in F which are of the form $x^\alpha e_j$ where $\deg e_j \leq d$ and are irreducible with respect to G . The irreducible terms in F_d of degree i are all $x^\alpha t^\beta e_j$ such that $\text{supp}(\alpha + \beta) \subseteq \text{ncrit } e_j$ and $\max \text{supp } \alpha \leq \min \text{supp } \beta$, and from this we can conclude that the map $x^\alpha t^\beta e_i \mapsto x^{\alpha+\beta} e_i$ is a bijection. \square

An immediate consequence of the preceding theorem is the following corollary that generalises [SV08], where it is shown that a homogeneous ideal with linear quotients is componentwise linear.

Corollary 4. *A graded module with initially linear syzygies is componentwise linear.*

Proof. Follows directly from Theorem 3 and Corollary 3. \square

4. A CONTRACTING HOMOTOPY

We will now consider the minimal resolution G_\bullet of a quadratic monomial ideal M with initially linear syzygies as a based complex of k -modules by the natural decomposition

$$G_n = \bigoplus_{\substack{\alpha, I, j \\ I \subseteq \text{crit } g_j}} k \cdot x^\alpha e_I g_j.$$

From a Morse matching on the digraph Γ_{G_\bullet} we will construct a contracting homotopy c on G_\bullet , that is a k -linear map satisfying

$$dc + cd = 1 - \eta.$$

The contracting homotopy will be used in the next section to show the existence of a DGA structure on G_\bullet in the case when M is a stable monomial ideal or a squarefree matroidal monomial ideal. The matching is constructed as follows. For $\beta \in \mathbb{N}^n$, let $V_{\beta,j}$ be the subset of V consisting of all $x^\alpha e_I g_j$ such that $\deg_{\mathbb{N}^n} x^\alpha e_I = \beta$. Exactly as in the proof of Lemma 1, we construct a partial matching $B_{\beta,j}$ on $\Gamma_{G_\bullet}|_{V_{\beta,j}}$

$$B_{\beta,j} = \{x^\alpha e_I g_j \rightarrow x^\alpha x_i e_{I \setminus i} g_j \mid i = \min(\text{supp } \alpha \cup I) \cap \text{crit } g_j\}.$$

Lemma 3. *The set $B = \bigcup_{\beta,j} B_{\beta,j}$ is a Morse matching on Γ_{G_\bullet} .*

Proof. The same argument as for Lemma 1 shows that each $B_{\beta,j}$ is a Morse matching on $\Gamma_{G_\bullet}|_{V_{\beta,j}}$, and we note that whenever there is an edge from a vertex in $V_{\beta,j}$ to a vertex in a different set $V_{\gamma,k}$ we have $x^\gamma e_k \prec x^\beta e_j$. \square

Let

$$\begin{aligned} \tilde{l}(\alpha, j) &= \min(\text{supp } \alpha \cap \text{crit}(g_j)), \\ c_0(x^\alpha \cdot e_I g_j) &= [\tilde{l}(\alpha, j) < \min I] \frac{x^\alpha}{x_{\tilde{l}(\alpha, j)}} e_{I \cup \tilde{l}(\alpha, j)} g_j. \end{aligned}$$

We can now define the map c by

$$c(x^\alpha \cdot e_I g_j) = c_0(x^\alpha \cdot e_I g_j) - c(dc_0(x^\alpha \cdot e_I g_j) - x^\alpha \cdot e_I g_j).$$

A consequence of the above lemma is:

Corollary 5. *The k -linear map c is a contracting homotopy on G_\bullet such that $\text{Im}(c)$ is spanned by the elements*

$$\{x^\alpha e_I g_j \mid \min((\text{supp } x^\alpha \cup I) \cap \text{crit } g_j) \in I\}.$$

Proof. This follows from [Skö06, Lemma 6]. \square

For monomial ideals with crit-monotone presentations, we can say a bit more about the contracting homotopy, by again using reductions for our description. We will define the set of c -critical indices of a basis element $e_I g_j$ by

$$c\text{-crit}(e_I g_j) = \{i \mid i \in \text{crit } g_j, i < \min I\}.$$

We have the following formula for the homotopy acting on a basis element:

$$c(x^\alpha e_I g_\beta) = \sum_{x^\delta e_J g_\gamma \in B^-} \sum_{p \in [x^\alpha e_I g_\beta \rightsquigarrow x^\delta e_J g_\gamma]} c_0 \rho_p(x^\alpha e_I g_\beta).$$

The composition with c_0 means that only elementary reduction paths $\alpha_0 \rightarrow \beta_1 \rightarrow \alpha_1$ where α_1 is in B^- will contribute to the result. These reduction paths are of the form

$$x^\alpha e_I g_\beta \rightarrow \frac{x^\alpha}{x_i} e_{I \cup \{i\}} g_\beta \rightarrow \frac{x^\alpha}{x_i} x^\delta e_I g_\gamma,$$

where $i = \min(c\text{-crit}(e_I g_\beta) \cap \text{supp } x^\alpha)$, $x^\delta g_\gamma = \text{nf}(x_i g_\beta)$, and $I \subseteq \text{crit}(g_\gamma)$. We can also note that for each k -basis element $x^\alpha e_I g_\beta$, there is at most one elementary reduction path emanating from it. This means that the terms that occur in $c(x^\alpha e_I g_\beta)$ are all of the form $\frac{x^\alpha}{m} n e_{i \cup I} g_\gamma$ where m divides x^α ,

We can now define a k -linear function ρ by setting

$$\rho(x^\alpha e_I g_\beta) = \begin{cases} \frac{x^\alpha}{x_i} x^\delta e_I g_\gamma, & \text{if } J \neq \emptyset \text{ with } i = \min J, \\ 0, & \text{if } J = \emptyset. \end{cases}$$

where $J = \text{supp } x^\alpha \cap c\text{-crit}(e_I g_\beta)$ and $\text{nf}(x_i g_\beta) = x^\delta g_\gamma$.

From these observations we can now deduce the following lemma.

Lemma 4. *Let M be a crit-monotone monomial ideal with initially linear syzygies. The contracting homotopy c is given by*

$$c(x^\alpha e_I g_\beta) = \sum_j c_0 \rho^j(x^\alpha e_I g_\beta),$$

where ρ is defined as above.

5. DGA STRUCTURES ON RESOLUTIONS

In this section we will construct a differential graded algebra structure on the minimal resolution of S/I where I is either a stable monomial ideal or a square-free matroidal ideal. We can thereby extend, with a simpler proof, the result of Peeva [Pee96] showing the existence of a DGA structure on the minimal resolution of S/I where I is a stable ideal.

Let \tilde{G}_\bullet be the resolution of S/I obtained by splicing the resolution $G_\bullet \rightarrow I$ with $0 \rightarrow I \rightarrow S \rightarrow S/I \rightarrow 0$. Thus, we have

$$\tilde{G}_\bullet : 0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow S \longrightarrow 0,$$

and we can extend the contracting homotopy c defined on G_\bullet to \tilde{G}_\bullet by setting $c(x^\alpha) = x^{\alpha-\beta} g_\beta$, where x^β is the smallest monomial generator of I with respect to \prec that divides x^α . It is easy to see that with this definition, $c^2(x^\alpha) = 0$, and therefore $c^2 = 0$.

We will now define a map $\mu : \tilde{G}_\bullet \otimes_S \tilde{G}_\bullet \rightarrow \tilde{G}_\bullet$ so μ is the multiplication in a DGA structure on \tilde{G}_\bullet . The technique we will use to establish this result rests upon the following Lemma, which is a special case of [ML63, Theorem IX.6.2].

Lemma 5. *Suppose that X_\bullet and Y_\bullet are complexes of S -modules, where $X_n = S \otimes_k V_n$ and $Y_n = S \otimes_k W_n$ for k -spaces V_n and W_n , $n \geq 0$. Furthermore, suppose that Y_\bullet is acyclic, with a contracting homotopy c satisfying $c^2 = 0$. Then, every S -linear map $\varphi_0 : X_0 \rightarrow Y_0$ has a unique lifting to a chain map $\varphi : X_\bullet \rightarrow Y_\bullet$ satisfying $\varphi(V_n) \subseteq \text{Im}(c)$. This map is defined inductively by*

$$\varphi_{n+1}(\bar{x}) = c\varphi_n d(\bar{x}), \quad \bar{x} \in V_{n+1}.$$

We call elements of $V_n \subseteq S \otimes_k V_n$ *reduced*; the reduced elements of \tilde{G}_\bullet are thus the S -basis elements of \tilde{G}_n .

Thus, we define our map μ on the reduced elements \bar{x} and \bar{y} of degrees m and n respectively by:

$$(3) \quad \mu(\bar{x} \otimes \bar{y}) = c\mu d(\bar{x} \otimes \bar{y}) = c\mu(d(\bar{x}) \otimes \bar{y}) + (-1)^m c\mu(\bar{x} \otimes d(\bar{y})).$$

Now, consider the composition

$$\tilde{G}_\bullet \xrightarrow{\simeq} S \otimes_S \tilde{G}_\bullet \xrightarrow{\iota \otimes 1} \tilde{G}_\bullet \otimes_S \tilde{G}_\bullet \xrightarrow{\mu} \tilde{G}_\bullet,$$

which is the identity in degree 0, and since $\mu(1 \otimes \bar{x}) \in \text{Im}(c)$; by Lemma 5, this is then the identity in all degrees, so $1 \in \tilde{G}_0$ is a multiplicative identity element.

Furthermore, letting τ be the twist morphism, $\tau(x \otimes y) = (-1)^{mn} y \otimes x$ where x and y are homogeneous of degrees m and n respectively, we have that μ and $\mu \circ \tau$ both are chain maps $\tilde{G}_\bullet \otimes \tilde{G}_\bullet \rightarrow \tilde{G}_\bullet$ that in degree 0 are given by $\mu(1 \otimes 1) = 1 = \mu \circ \tau(1 \otimes 1)$. Since for reduced elements \bar{x} and \bar{y} we have that $\mu \circ \tau(\bar{x} \otimes \bar{y}) \in \text{Im}(c)$, Lemma 5 gives that $\mu = \mu \circ \tau$, so μ is graded commutative. Thus, to show that μ gives a DGA structure to \tilde{G}_\bullet , it remains to show that μ is associative, that is, that $\mu(1 \otimes \mu) = \mu(\mu \otimes 1)$.

Recall that for a basis element e_{Ig_j} we have $c\text{-crit}(e_{Ig_j}) = \{i \mid i \in \text{crit } g_j, i < \min I\}$, and we now extend this to the whole of \tilde{G}_\bullet by letting

$$c\text{-crit}\left(\sum_i p_i \cdot e_{I_i g_{j_i}}\right) = \bigcup_i c\text{-crit}(e_{I_i g_{j_i}})$$

where we have no redundant terms in the sum. We are now in a position to formulate and prove the lemma that we will use to show associativity. We will in the following write $x \star y$ for $\mu(x \otimes y)$.

Lemma 6. *If, for all basis elements e_{Ig_i}, e_{Jg_j} , we have*

$$(4) \quad c\text{-crit}(e_{Ig_i} \star e_{Jg_j}) \subseteq c\text{-crit}(e_{Ig_i}) \cap c\text{-crit}(e_{Jg_j}),$$

then \star is associative.

Proof. Since $1 \star x = x$ for all $x \in \tilde{G}_\bullet$, we only have to show that

$$(5) \quad e_{Ig_i} \star (e_{Jg_j} \star e_{Kg_k}) = (e_{Ig_i} \star e_{Jg_j}) \star e_{Kg_k}$$

for all basis elements e_{Ig_i}, e_{Jg_j} and e_{Kg_k} , and this now follows if we can show that

$$(6) \quad e_{Ig_i} \star (e_{Jg_j} \star e_{Kg_k}) \in \text{Im } c,$$

holds for all e_{Ig_i}, e_{Jg_j} and e_{Kg_k} , since then by the graded commutativity of \star we would also get that

$$(7) \quad (e_{Ig_i} \star e_{Jg_j}) \star e_{Kg_k} \in \text{Im } c$$

and by Lemma 5 we can conclude that they are equal. Now, suppose that $x^\alpha e_{Lg_l}$ occurs as a term in $e_{Jg_j} \star e_{Kg_k}$, then, by the condition of the lemma, no variable occurring in x^α will be c -critical in $e_{Ig_i} \star e_{Lg_l}$, and thus (6) follows, and the proof is complete. \square

We will now in a series of lemmata show that the conditions of Lemma 6 are satisfied for the minimal resolution of stable and squarefree matroidal ideals. We start with the case where one of the basis elements have minimal degree.

Lemma 7. *Let g_α and $e_I g_\beta$ be basis elements in the minimal resolution of a stable ideal. We then have*

$$c\text{-crit}(c(x^\alpha e_I g_\beta)) \subseteq c\text{-crit}(g_\alpha).$$

Proof. By Lemma 4, we have that

$$c(x^\alpha e_I g_\beta) = \sum_j c_0(x^{\alpha_j} e_I g_{\beta_j})$$

where α_j and β_j satisfy $x^{\alpha_j} = x^\alpha v_j / u_j$ and $v_j g_{\beta_j} = \text{nf}(u_j g_\beta)$ for some monomial u_j of degree j dividing x^α . Now, if $c_0(x^{\alpha_j} e_I g_{\beta_j}) \neq 0$, then by the crit-monotonicity of the stable ideals, $c_0(x^{\alpha_j} e_I g_{\beta_j}) = x^{\alpha_j} / x_k e_{I \cup k} g_{\beta_j}$, for a $k \in \text{supp}(x^\alpha / u_j)$, and thus

$$c\text{-crit}(x^{\alpha_j} / x_k e_{I \cup k} g_{\beta_j}) = [1, k-1] \subseteq \text{crit}(g_\alpha).$$

□

Before showing the corresponding result for squarefree matroidal ideals, we have to say something about the term order we use.

Lemma 8. *Let M be a squarefree matroidal ideal in S , then M is shellable with respect to a lexicographic ordering.*

Proof. Essentially the same as the proof given by Herzog and Takayama [HT02, Lemma 1.3] for the revlex order. □

From this we can conclude that the order given by $m g_\alpha \prec n g_\beta$ whenever $x^\alpha <_{\text{lex}} x^\beta$ gives us initially linear syzygies.

Lemma 9. *Let g_α and $e_I g_\beta$ be basis elements in the minimal resolution of a square-free matroidal ideal. We then have*

$$c\text{-crit}(c(x^\alpha e_I g_\beta)) \subseteq c\text{-crit}(g_\alpha).$$

Proof. Again, by Lemma 4, we have that

$$c(x^\alpha e_I g_\beta) = \sum_i c_0(x^{\alpha_i} e_I g_{\beta_i})$$

where α_j and β_j satisfy $x^{\alpha_j} = x^\alpha v_j / u_j$ and $v_j g_{\beta_j} = \text{nf}(u_j g_\beta)$ for some monomial u_j of degree j dividing x^α . Now assume that $j \in \text{crit}(g_{\beta_k})$ for some j such that j is less than all elements in $\text{crit}(g_{\beta_k}) \cap \text{supp} \frac{x^\alpha v_k}{u_k}$. From the observation that $x^{\alpha_i} x^{\beta_i} = x^\alpha x^\beta$ for all i we can conclude that

$$\text{nf}(x_j x^{\alpha_k} g_{\beta_k}) = \text{nf}(x_j x^\alpha g_\beta).$$

We can reduce $x_j x^{\alpha_k} g_{\beta_k}$ to $x_l x^{\alpha_k} g_{\beta'_k}$ for some $l > j$, and there cannot be any later reduction of the form $x_m g_\sigma \rightarrow x_j g_{\sigma'}$ in a chain of reductions starting in $x_l x^{\alpha_k} g_{\beta'_k}$, since that would imply that $m < j$, and that $x_m \in \text{crit}(g_{\beta_k})$, and then we would have $m \in \text{supp} x^{\alpha_k}$ which contradicts the choice of j . Thus we have for $x^\sigma g_\tau = \text{nf}(x_j x^\beta g_\alpha)$ that $j \in \text{supp} \tau$, and thus, by the crit-monotonicity, $j \in \text{crit}(g_\alpha)$. □

Lemma 10. *Let g_α and $e_I g_\beta$ be basis elements in the minimal resolution of a stable ideal or a squarefree matroidal ideal. We then have*

$$c\text{-crit}(c(x^\alpha e_I g_\beta)) \subseteq c\text{-crit}(e_I g_\beta).$$

Proof. By Lemma 4, we can see that the elements that appear in $c(x^\alpha e_I g_\beta)$ are of the form $x^\gamma e_{i \cup I} g_\delta$, and by the crit-monotonicity we know that $\text{crit}(g_\delta) \subseteq \text{crit}(g_\beta)$, hence the statement follows. □

We now turn to the case where the first basis elements in the product has non-minimal degree.

Lemma 11. *Let $e_I g_\alpha$ and $e_J g_\beta$ be two basis elements in \tilde{G}_\bullet with $I \neq \emptyset$. If the inclusion*

$$c\text{-crit}(e_K g_\gamma \star e_L g_\delta) \subseteq c\text{-crit}(e_K g_\gamma) \cap c\text{-crit}(e_L g_\delta),$$

holds for all pairs of basis elements $e_K g_\gamma$ and $e_L g_\delta$ where $|K| + |L| < |I| + |J|$, then

$$c\text{-crit}(c(d(e_I g_\alpha) \star e_J g_\beta)) \subseteq c\text{-crit}(e_I g_\alpha) \cap c\text{-crit}(e_J g_\beta).$$

Proof. By the crit-monotonicity, the differential d can be written as

$$d = \sum_j (-1)^{j-1} d_j^L - \sum_j (-1)^{j-1} d_j^R,$$

so we consider the terms $c(d_i^L(e_I g_\alpha) \star e_J g_\beta)$ and $c(d_i^R(e_I g_\alpha) \star e_J g_\beta)$ separately.

If $i = 1$, we have

$$c(d_1^L(e_I g_\alpha) \star e_J g_\beta) = c(x_i e_{I \setminus i} g_\alpha \star e_J g_\beta), \quad i = \min I.$$

Suppose that $m e_K g_\gamma$ is a term in the product $e_{I \setminus i} g_\alpha \star e_J g_\beta$; by assumption, we then have that $c\text{-crit}(e_K g_\gamma) \subseteq c\text{-crit}(e_{I \setminus i} g_\alpha)$ and that $\text{supp } m \cap c\text{-crit}(e_K g_\gamma) = \emptyset$. If $c(x_i m e_K g_\gamma)$ is nonzero, then, by Lemma 4 we get that

$$c(x_i m e_K g_\gamma) = m e_{i \cup K} g_\gamma$$

and thus,

$$\begin{aligned} c\text{-crit}(c(x_i m e_K g_\gamma)) &= c\text{-crit}(e_{i \cup K} g_\gamma) \\ &= c\text{-crit}(e_K g_\gamma) \cap [1, i-1] \\ &\subseteq c\text{-crit}(e_{I \setminus i} g_\alpha) \cap c\text{-crit}(e_J g_\beta) \cap [1, i-1] \\ &= c\text{-crit}(e_I g_\alpha) \cap c\text{-crit}(e_J g_\beta). \end{aligned}$$

If $i > 1$, then we have

$$c(d_i^L(e_I g_\alpha) \star e_J g_\beta) = c(x_i e_{I \setminus i} g_\alpha \star e_J g_\beta), \quad i > \min I.$$

If $m e_K g_\gamma$ occurs in the product $e_{I \setminus i} g_\alpha \star e_J g_\beta$, then, since $i \notin c\text{-crit}(e_{I \setminus i} g_\alpha)$, the hypothesis of the lemma gives us that $i \notin c\text{-crit}(e_K g_\gamma)$, and therefore $c(x_i m e_K g_\gamma) = 0$.

Now, we look at d_i^R ; for all i we have

$$c(d_i^R(e_I g_\alpha) \star e_J g_\beta) = c(n e_{I \setminus i} g_\gamma \star e_J g_\beta)$$

for some monomial n with $\text{supp } n \cap c\text{-crit}(e_{I \setminus i} g_\gamma) = \emptyset$. Let $m e_K g_\delta$ be a term in the product $e_{I \setminus i} g_\gamma \star e_J g_\beta$, by assumption $c\text{-crit}(m e_K g_\delta) \subseteq c\text{-crit}(e_{I \setminus i} g_\gamma)$ and since $\text{supp } m \cap c\text{-crit}(e_K g_\gamma) = \emptyset$ and $\text{supp } n \cap c\text{-crit}(e_K g_\gamma) = \emptyset$, we can conclude that $c(mn e_K g_\gamma) = 0$. \square

Theorem 4. *The minimal resolution of M where M is a squarefree matroidal ideal or a stable ideal has a DGA structure.*

Proof. By Lemma 6 it suffices to show that

$$c\text{-crit}(e_I g_\alpha \star e_J g_\beta) \subseteq c\text{-crit}(e_I g_\alpha) \cap c\text{-crit}(e_J g_\beta)$$

holds for all basis elements $e_I g_\alpha$ and $e_J g_\beta$, and by the definition of the product, we thus need to verify the relations

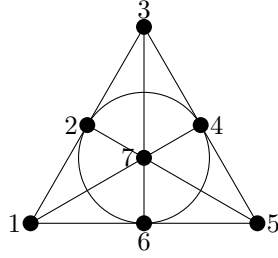
$$(8) \quad c\text{-crit}(c(d(e_I g_\alpha) \star e_J g_\beta)) \subseteq c\text{-crit}(e_I g_\alpha) \cap c\text{-crit}(e_J g_\beta),$$

$$(9) \quad c\text{-crit}(c(d(e_J g_\beta) \star e_I g_\alpha)) \subseteq c\text{-crit}(e_I g_\alpha) \cap c\text{-crit}(e_J g_\beta).$$

We proceed by induction on $|I| + |J|$. If $|I| = |J| = 0$, (8) and (9) hold by Lemmas 7, 9 and 10. Now for the case $|I| + |J| > 0$, the inclusion (8) holds if $|I| = 0$ by Lemmas 7, 9 and 10, and by induction and Lemma 11 if $|I| > 0$; and similarly for the inclusion (9). \square

We will finish the paper by looking at the multiplicative structure of the minimal resolution of a small squarefree matroidal ideal.

Example 1. The Fano matroid is the matroid on the ground set $\{1, 2, \dots, 7\}$, where every 3-element set is a basis, except for the following sets: $\{1, 2, 3\}$, $\{1, 4, 7\}$, $\{1, 5, 6\}$, $\{2, 4, 6\}$, $\{2, 5, 7\}$, $\{3, 4, 5\}$, $\{3, 6, 7\}$. The Fano matroid is often visualised using the following diagram



where a curve is drawn through every 3-element circuit. Let I be the ideal in $S = k[x_1, \dots, x_7]$ generated by the monomials corresponding to the bases in the Fano matroid. This ideal then has $\binom{7}{3} - 7 = 28$ generators, so for space reasons we will not describe the full multiplication table on the minimal resolution of S/I . Instead we will look at the resolution of S'/J , where the ideal J is generated by the monomials in I whose support is contained in $\{1, 2, 3, 4\}$ and S' is the polynomial ring $k[x_1, \dots, x_4]$. This is then going to be a subalgebra of the minimal resolution of S/I . Thus, $J = (x_1 x_2 x_4, x_1 x_3 x_4, x_2 x_3 x_4)$, and we have the following basis elements in the resolution:

Degree	Basis elements
0	1
1	$g_{124}, g_{134}, g_{234}$
2	$e_2 g_{134}, e_1 g_{234}$

Most of the products are either zero for degree reasons, or trivial due to multiplication by the identity element, so we are left with the following products to calculate: $g_{124} \star g_{134}$, $g_{124} \star g_{234}$ and $g_{134} \star g_{234}$:

$$\begin{aligned}
 g_{124} \star g_{134} &= c(x_1 x_2 x_4 g_{134}) - c(x_1 x_3 x_4 g_{124}) \\
 &= x_1 x_4 e_2 g_{134} - 0 \\
 &= x_1 x_4 e_2 g_{134},
 \end{aligned}$$

$$\begin{aligned}
g_{124} \star g_{234} &= c(x_1 x_2 x_4 g_{234}) - c(x_2 x_3 x_4 g_{124}) \\
&= x_2 x_4 e_1 g_{234} - 0 \\
&= x_2 x_4 e_1 g_{234}, \\
g_{134} \star g_{234} &= c(x_1 x_3 x_4 g_{234}) - c(x_2 x_3 x_4 g_{134}) \\
&= x_3 x_4 e_1 g_{234} - x_3 x_4 e_2 g_{134}.
\end{aligned}$$

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